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AUTHORS' CLOSURE

The formulation in our paper is based on the energy dissipation during the curvature jump process at the moving hinge. It has been shown that the extension of a rigid–perfectly plastic (RPP) solution to a rigid–strain hardening (RSH) material is no more complex than the RPP solution itself, provided that the kinematic behaviour of the moving hinge is properly understood. The extra care one must take is that, for a RSH material, the deformation history should be monitored at every step throughout the entire process. In principle, if the material behaviour is given and the deformation mode is known, the applied load can always be calculated by the energy conservation principle, in a differential (continuous) or integral (discrete) manner.

If the curvature change, from K^+ to K^-, is known from a particular deformation mode, linearity of the moment function results because of the adopted linearly hardening material [see Fig. 3(b) from the original paper or Fig. 1 from the Letter to the Editor]. If the change of curvature leads to a reversed loading, the moment will reverse in an abrupt manner before the hardening process begins [Fig. 3(c) from the original paper]. Therefore, the dashed line in Fig. 1 from the Letter to the Editor is not a linear approximation, *per se*; it is dictated by the material behaviour.

It is apparent that the assumption of $M = M(K^+)$ for the hinge moment is not just an approximation; it is conceptually flawed. In our analysis, the energy dissipation is calculated for the complete jump process according to the material law. This does not automatically lead to eqn (4) from the Letter to the Editor, as suggested by the reader. Equation (4) from the Letter to the Editor is only a particular case for a positive jump at the hinge, as shown in Fig. 3(b) from the original paper. Note that $M(K^+)$ is the moment before the jump, M_i. If the moment M_p jumps to $-M_p$, as shown in Fig. 2(c) from the original paper, M would become zero according to eqn (4) from the Letter to the Editor.

An appropriate form of the energy rate, for a RPP material, during the jump process is

$$\dot{E}_H = |M_p \dot{H}[K]| \tag{1}$$

or, if it is expanded

$$\dot{E}_H = M_p \dot{H}[K]^+, \tag{2a}$$

$$\dot{E}_H = (-M_p)\dot{H}[K]^-, \tag{2b}$$

where $[\ ]^+$ and $[\ ]^-$ represent, respectively, a positive and negative curvature jump. Equation (1) is equivalent to eqn (9) in the original paper. It should be noted that, in a RPP material, M is not necessarily equal to M_p, as seen from eqns (2a, b) and Fig. 2 from the original paper. For a RSH material, more complicated equations result. Compare, for example, eqn (11) with eqns (16), (17) and (18), all from the original paper.

In both cases, i.e. RPP and RSH materials, there may be a moment jump across the hinge depending on the nature of the curvature jump (see Figs 2, 3 and 4 in the original paper). In fact, the bending moment is nonexistent at the hinge; it belongs to the neighbouring arcs. A hinge is merely an intersection of two arcs, not a material point that can admit any load. Unlike the conventional plastic hinge, the moving hinge in the present context does not have the moment continuity at the hinge location. Plastic hinge travel is a real physical phenomenon (Hopkins, 1955) while the moving hinge is a theoretical concept only to facilitate describing structural behaviour. Note the statement made by Hopkins

(1955) on page 42: "the bending moment M and the shear force Q are continuous across H (a yield hinge)."

Smearing out a plastic hinge into a finite region by the existence of strain hardening occurs only in a conventional (and real) plastic hinge, not in a moving hinge. A moving hinge is simply a *curvature discontinuity* whereas a real plastic hinge has an *infinite curvature* in the neighbourhood. Strain hardening renders that infinite curvature finite, resulting in a finite deformation zone. Since a "hinge" with a finite length is no longer a hinge, a plastic hinge exists only in a non-hardening material. The concept of a finite hardening zone has been discussed earlier in connection with finite deflection problems (Sherbourne and Lu, 1993).

In tackling buckle propagation problems using a similar ring-crushing model, Croll (1985) chose three fixed arc lengths, thus beautifully averting the moving hinge phenomenon. The continuous model by Croll is only a part of the moving hinge approach. The application of Croll's model is limited in the sense that the degree of freedom is constricted in modelling a deformation mode. Interestingly, there is one advantage in this simpler mode. That is, the Ramberg–Osgood relation can be used, which the moving hinge method cannot seem to accommodate. The moving hinge method does not allow elasticity to be considered.

In the work by Wierzbicki and Bhat (1986), three arc lengths vary with the progression of deformation while the three corresponding curvatures are assumed to remain constant. Therefore, the plastic energy is dissipated only in the moving hinge travel or curvature jump. The three arcs remain essentially rigid—shrinking or extending, depending on the direction of the hinge movement.

In the present analysis, both continuous curvature change and curvature jump at the moving hinge are included. This more realistic approach will probably enhance the results to a certain degree.

The moving hinge technique is one form of the discretization process to approximate the actually-continuous curvature change, resulting in a closed-form solution. It must be pointed out that, at the moving hinge, the equilibrium condition is not satisfied, due to the moment discontinuity. However, a similar feature can be found in other discretization methods, say, finite element analysis. The moving hinge technique is a powerful tool in assessing the global, or macro, structural behaviour, rather than the micro behaviour in the continuum sense, since the total energy balance is maintained.

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